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Titel: $A_0 = dr(N)$

$N = pA + qB + rC + sD$ · $p \equiv 1 \pmod q$ · Exact O(1) formulas

A complete theory for linear Diophantine representation counting across 2D, 3D, and 4D systems. The minimal A-coefficient always equals the digital root of N — proven for all coprime pairs p ≡ 1 (mod q). Verified for 12 systems. No recursion. No approximation.

The 2D System

$N = pA + qB \cdot p \equiv 1 \pmod q$

From digital roots to exact O(1) counting — the complete 2D theory

§0 — What is a representation?

A **representation** of N in the system (p, q) is a pair of non-negative whole numbers (A, B) such that pA + qB = N.

Think of it this way: you have only banknotes of €19 and €9. In how many ways can you pay exactly €958? That is exactly the question this system

answers. The answer turns out to be 6 — and you can find them all in a single calculation, without trying every combination.

KEY CONCEPTS

Concept	Meaning	Example
$\lfloor x \rfloor$ (floor)	Round down to nearest integer	$\lfloor 7.8 \rfloor = 7$, $\lfloor 50/9 \rfloor = 5$
mod (modulo)	Remainder after division	17 mod 9 = 8
$p \equiv 1 \pmod q$	p divided by q leaves remainder 1	$19 \equiv 1 \pmod 9$
Digital root	Add digits until one digit remains	958→22→4, dr(958)=4
O(1)	Constant time regardless of N	10 steps for $N=10^{100}$

§1

The Main Theorem: A_0 = dr(N)

KEY INSIGHT — WHY $P \equiv 1 \pmod Q$?

Take p=19, q=9: $19 = 2 \times 9 + 1$, so $19 \equiv 1 \pmod 9$. This has a powerful consequence:

$$N = 19A + 9B \rightarrow N \bmod 9 = (19A + 9B) \bmod 9 = (1 \cdot A + 0 \cdot B) \bmod 9 = A \bmod 9$$

B disappears completely from the equation mod q! Therefore $A_0 = N \bmod q$ — found in one step.

THEOREM 2.2 — MAIN THEOREM

$$A_0 = \mathrm{dr}_q(N) \quad \text{for all } N \geq pq$$

The smallest A for which N = pA + qB has a non-negative solution is the digital root of N. All solutions form a regular arithmetic ladder:

$$A = A_0 + q \cdot k, \quad B = \frac{N - p \cdot A}{q} - p \cdot k \quad \text{for } k = 0, 1, \ldots, k_{max}$$

where $A_0 = N \bmod q$. Each step increases A by q=9 and decreases B by p=19.

PROOF — 3 STEPS

Step 1: Take N = pA + qB modulo q. Since q·B $\equiv$ 0 (mod q) and p $\equiv$ 1 (mod q): N $\equiv$ A (mod q). So A must be congruent to N mod q.

Step 2: The smallest non-negative A with this congruence is A_0 = N mod q. Since N $\geq$ pq, B = $(N - p \cdot A_0)/q \geq 0$. So (A_0, B) is a valid representation.

Step 3: All other representations come from A → A+q, B → B−p. This preserves pA + qB = N. We stop when B would become negative. ■

All 6 Representations of N = 958 in System (19, 9)

#	A	B	19×A	9×B	Sum	DR(A)	DR(B)	Remark
1	4	98	76	882	958	4	8	A_0 = dr(958) = 4 ← start
2	13	79	247	711	958	4	7	A increases by 9 each row
3	22	60	418	540	958	4	6	S(958) = 22 → valid A
4	31	41	589	369	958	4	5	589 = 19 × 31
★ 5	**40**	**22**	**760**	**198**	**958**	**4**	**4**	**★ Symmetry point: dr(A)=dr(B)=dr(N)**

6	49	3	931	27	958	4	3	Last before B goes negative

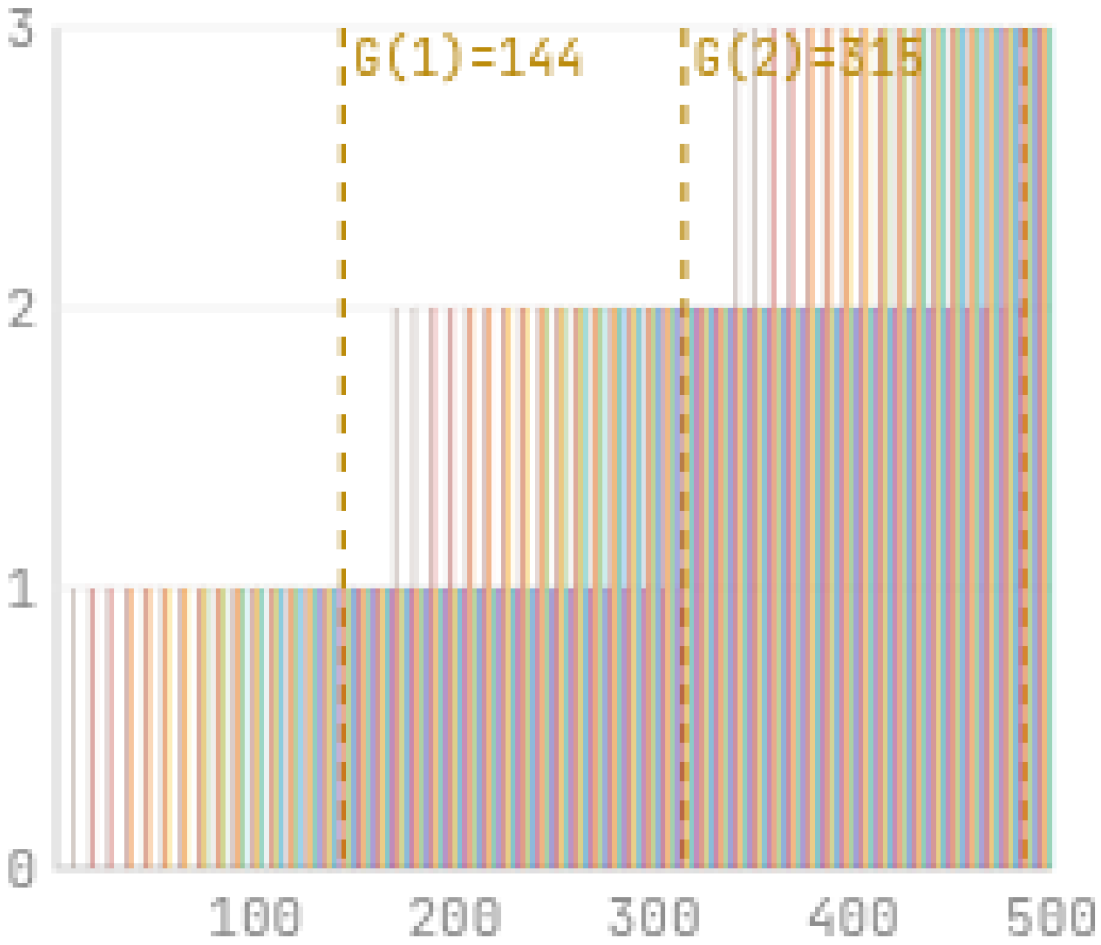

Figure 1 — R(N) for N = 1 to 500 · System (19,9) · Coloured by digital root (1–9). Each colour is one residue class. Dashed lines = family boundaries G(k) = 171k − 27. The bars grow roughly linearly; every 171 units the minimum rises by 1.

§2 — The Closed Formula for R(p, q; N)

THEOREM 3.1 — REPRESENTATION FORMULA

$$R(p, q; N) = \left\lfloor \frac{\lfloor N/p \rfloor - A_0}{q} \right\rfloor + 1$$

This formula is **exact** — not an approximation. Two operations: (1) compute $A_0 = N \bmod q$, (2) apply the floor formula. That is O(1).

Asymptotically: $R(N) \sim N/(p \cdot q)$ for large N.

N	Digit Chain	⌊N/19⌋	A_0 = N mod 9	Calculation	R(N)
366	366 → 15 → 6	19	6	⌊(19−6)/9⌋+1	2
592	592 → 16 → 7	31	7	⌊(31−7)/9⌋+1	3
589	589 → 22 → 4	31	4	⌊(31−4)/9⌋+1	4
958	**958 → 22 → 4**	**50**	**4**	**⌊(50−4)/9⌋+1**	**6**
2334	2334 → 12 → 3	122	3	⌊(122−3)/9⌋+1	14
6666	6666 → 24 → 6	350	6	⌊(350−6)/9⌋+1	39

§3 — Threshold Values and Family Boundaries

For a fixed digital root r, the t-th threshold is $T(t, r) = 19r + 171(t-1)$. Every time the minimum number of representations increases by 1, the boundary shifts by exactly $171 = 19 \times 9 = p \times q$.

$$G(k) = 171k - 27 \quad \text{(for the (19,9)-system)}$$

$G(1) = 144$ ← Frobenius boundary: largest non-representable number is 143

$G(10) = 1683$ ← from here every N has at least 10 representations

t	DR 1	DR 2	DR 3	DR 4	DR 5	DR 6	DR 7	DR 8	DR 9
1	19	38	57	76	95	114	133	152	9
2	190	209	228	247	266	285	304	323	171
3	361	380	399	418	437	456	475	494	342
4	**532**	**551**	**570**	**589**	**608**	**627**	**646**	**665**	**513**
5	703	722	741	760	779	798	817	836	684
6	874	893	912	931	950	969	988	1007	855
10	1558	1577	1596	1615	1634	1653	1672	—	1539

§4 — Symmetry Points

DEFINITION 5.1 — SYMMETRY POINT

A representation (A, B) of N is a **symmetry point** if dr(A) = dr(B) = dr(N). All three numbers share the same digital root.

N	DR(N)	Symmetry Point(s)	Remark
958	4	(40, 22)	Representation 5 of 6
1680	6	(60, 60)	Unique: A = B
6666	6	(24, 690)	24×19+690×9=6666 ✓
2520	9	(9, 261) and (90, 90)	Two symmetry points

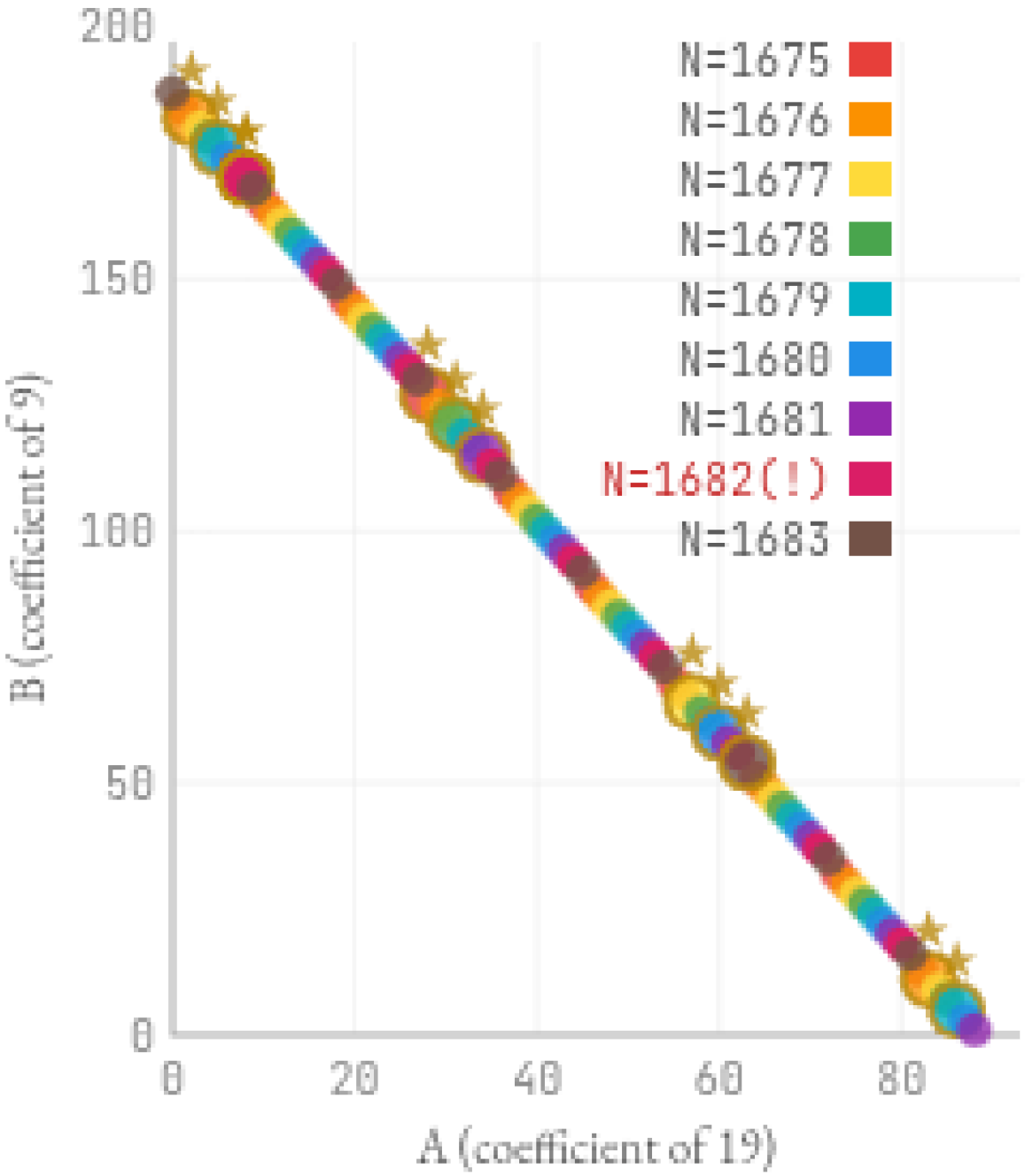

Figure 2 — All (A,B)-pairs for the complete family N = 1675–1683. Each point is one representation. Gold stars ★ = symmetry points where dr(A) = dr(B) = dr(N). All representations lie on lines of slope −19/9. N = 1682 (red) has only 9 points — the single gap in the otherwise perfect family.

§5 — The Reduction Chain Principle

The reduction chain of N yields two mandatory A-coordinates: the intermediate stop S(N) (first digit sum) and the digital root dr(N). This holds for *every* number, without exception.

> **WHY THIS WORKS**
>
> All A-values share the same remainder mod 9 as N (since A ≡ N (mod 9)). And S(m) ≡ m (mod 9) for any m — because 10 ≡ 1 (mod 9), so adding digits does not change the remainder mod 9. Therefore every value in the chain N → S(N) → ··· → dr(N) falls in the A-row.

N	Chain	Via S(N)	Via dr(N)
366	366 → 15 → 6	A=15: 15×19+9×9=366 ✓	A=6: 6×19+28×9=366 ✓
592	592 → 16 → 7	A=16: 16×19+32×9=592 ✓	A=7: 7×19+51×9=592 ✓
958	**958 → 22 → 4**	**A=22: 22×19+60×9=958** ✓	**A=4: 4×19+98×9=958** ✓
19999	19999 → 37 → 10 → 1	A=37: 37×19+2144×9=19999 ✓	A=1: 1×19+2220×9=19999 ✓

§6 — Additivity Theory

THEOREM 7.1 — ADDITIVITY AT THREE LEVELS SIMULTANEOUSLY

Let a + b = c. If (A_1, B_1) is a representation of a and (A_2, B_2) of b, then (A_1+A_2, B_1+B_2) is a valid representation of c.

Proof: $p(A_1+A_2) + q(B_1+B_2) = pA_1+qB_1 + pA_2+qB_2 = a+b = c.$ ■

This works simultaneously at three levels: (1) numbers, (2) coordinates, (3) digital roots: $dr(a)+dr(b) \equiv dr(c) \pmod 9$.

STRUCTURAL TRIANGLE: 366 + 592 = 958

Level	366	+	592	=	958
Numbers	366	+	592	=	958 ✓
Min. rep.	(6, 28)	+	(7, 51)	=	(13, 79) → 13×19+79×9=958 ✓
Alt. rep.	(15, 9)	+	(16, 32)	=	(31, 41) → 31×19+41×9=958 ✓
Digital roots	**dr=6**	**+**	**dr=7**	**=**	**13 → dr=4 = dr(958)** ✓

366 is the only number whose minimal pair (6, 28) consists of the first two perfect numbers.

Moreover 19 + 9 = 28 — the two system parameters sum to the second perfect number.

§7 — Matryoshka Theory

THEOREM 8.1 — MATRYOSHKA THEORY (COMPLETE PROOF)

Let $N \geq pq$ and let $N \to c_1 \to c_2 \to \cdots \to c_k = dr(N)$ be the complete reduction chain. Then for each c in the chain: c is an A-coordinate in the representations of N, provided $N \geq p \cdot c$.

Step 1: At every step $c \equiv N \pmod{q}$. Base case: $N \equiv N$. Inductive: $S(m) \equiv m \pmod{9}$ since $10 \equiv 1 \pmod{9}$. ✓

Step 2: Set $A = c$, then $B = (N - p \cdot c)/q$. From step 1: $N - p \cdot c \equiv 0 \pmod{q}$, so B is an integer. ✓

Step 3: $p \cdot c + q \cdot B = N$. ✓So (c, B) is a valid representation.

Step 4: $c \equiv A_0 \pmod{q}$, so $c = A_0 + k \cdot q$ for some $k \geq 0$ — it is in the standard A-ladder. ✓■

ENTROPY COROLLARY 8.2

$dr(N)$ is the most compressed representation of N — minimal information, but always recoverable as the smallest A-coordinate. No information is lost: it is redistributed over the A-coordinates.

§8 — Perfect Numbers in the 19-9 System

THEOREM 10.1 — PERFECT NUMBER EMBEDDING

Every even perfect number $P \geq 28$ satisfies $P \equiv 1 \pmod 9$. Therefore $dr_9(P) = 1$ and $A_0 = 1$. Moreover S(P) is always a valid A-coordinate when $P \geq 19 \cdot S(P)$.

Perfect Number P	S(P) = A	Chain of A	Verification
28	10	10 → 1	1×19+1×9=28 — unique ✓
496	19	19 → 10 → 1	19×19+15×9=361+135=496 ✓
8128	19	19 → 10 → 1	19×19+863×9=361+7767=8128 ✓
33,550,336	28	28 → 10 → 1	S(P)=28 → chain leads to dr=1 ✓

§9 — The Complete Family: N = 1675–1683

The nine consecutive integers 1675–1683 form the first contiguous sequence where each of the nine digital roots 1–9 appears exactly once and every member has reached its full representation count.
From N = 1683 onward, every integer has at least 10 representations.

REMARKABLE EXCEPTION

N = 1682 has only 9 representations, because digital root 8 reaches the boundary one step later than the others. This is the only "gap" in the otherwise perfect family. N = 1683 has $A_0 = 0$ (since 1683 = 9 × 187), closing the family.

N	Chain	DR	A_0	R(N)	Symmetry Point
1675	1675 → 19 → 10 → 1	1	1	10	(28, 127)

1676	1676 → 20 → 2	2	2	10	(2,182) and (83,11)
1677	1677 → 21 → 3	3	3	10	(57, 66)
1678	1678 → 22 → 4	4	4	10	(31, 121)
1679	1679 → 23 → 5	5	5	10	(5,176) and (86,5)
1680	1680 → 15 → 6	6	6	10	(60, 60) ← unique A=B
1681	1681 → 16 → 7	7	7	10	(34, 115)
1682	**1682 → 17 → 8**	**8**	**8**	**9 (!)**	**(8, 170) — the gap**
1683	1683 → 18 → 9	9	0	10	(63, 54)

PART 2

The 3D System & A-C Coupling

$$N = pA + qB + rC$$

Where recursion ends and periodicity begins

§0 — What Changes in Three Dimensions?

In 2D we had two variables and one equation. In 3D we add a third variable multiplied by r. The classical approach applies the 2D formula for every possible value of C — from C=0 to ⌊N/r⌋.

N	Classical Approach (3D)	This Paper — O(1)	Speed-up
1,000	~333 steps	10 steps	33×
1,000,000	~333,000 steps	10 steps	33,300×
10^9	~333,333,333 steps	10 steps	33,333,333×
10^{100}	3.3×10^{97} steps	10 steps	Unbounded

§1 — The A-C Coupling

THE KEY INSIGHT — COUPLING

When $p \equiv 1 \pmod q$ and r divides q, take $N = pA + qB + rC$ modulo q. The term qB vanishes, p simplifies to 1, and we get:

$$A + rC \equiv N \pmod q$$

A and C are not independent — they are locked together. Once A is fixed, C is constrained. Once C is fixed, A is constrained. The only truly free variable is B. This collapses the 2D search over (A, C) into a 1D problem.

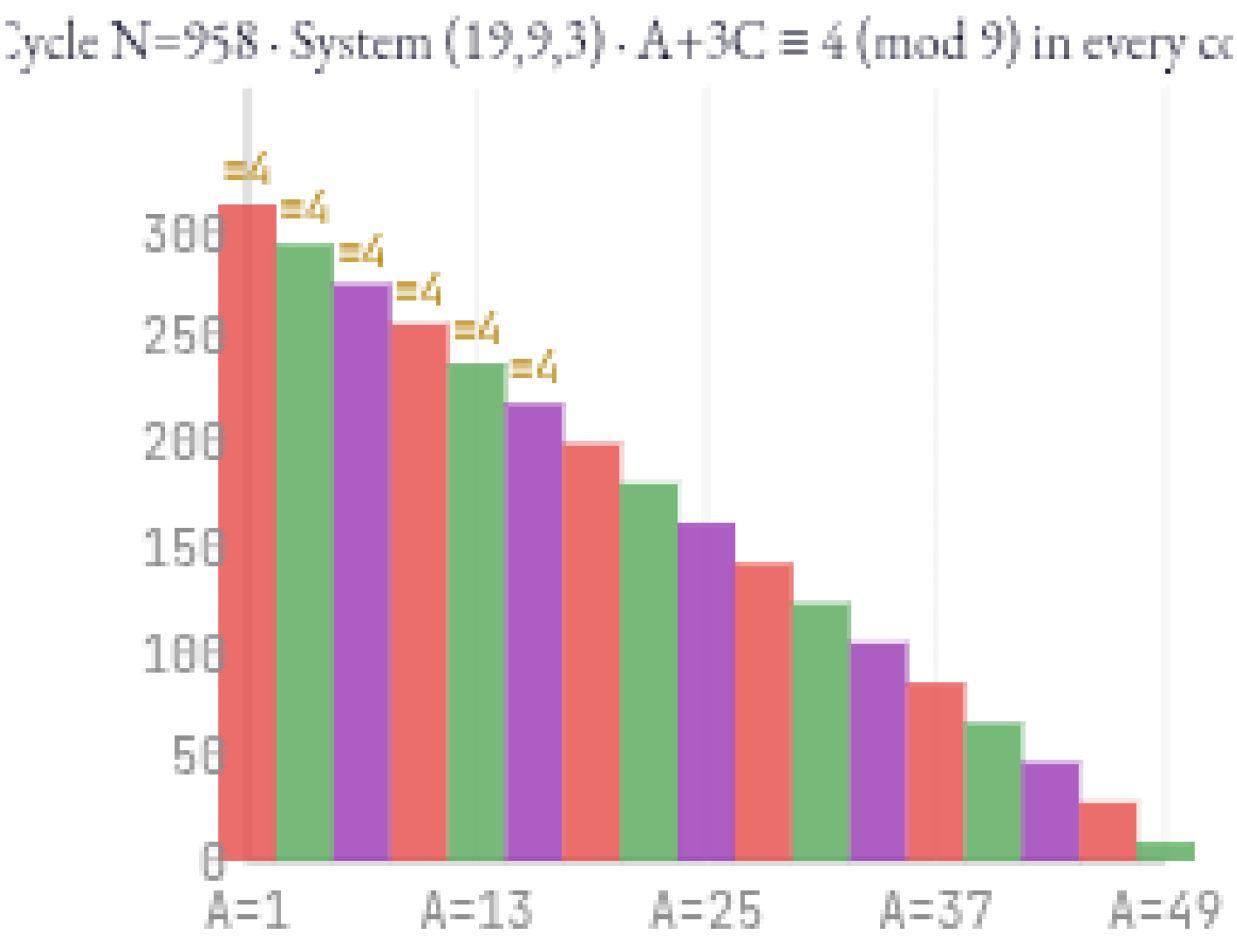

Figure 3 — A-C Coupling Cycle for N = 958 · System (19,9,3) · Period = 3. In every row A + 3C mod 9 = 4 = dr(958), without exception. The coupling invariant made visible.

PERIOD TABLE — 8 VERIFIED SYSTEMS

System	$p \equiv 1 \pmod{q}$	$r \mid q$	$p \equiv 1 \pmod{r}$	Period = q/r
(19,9,3)	19 = 2×9+1 ✓	3|9 ✓	19 = 6×3+1 ✓	**3**
(10,9,3)	10 = 1×9+1 ✓	3|9 ✓	10 = 3×3+1 ✓	3
(37,9,3)	37 = 4×9+1 ✓	3|9 ✓	37 = 12×3+1 ✓	3
(7,6,3)	7 = 1×6+1 ✓	3|6 ✓	7 = 2×3+1 ✓	2
(7,6,2)	7 = 1×6+1 ✓	2|6 ✓	7 = 3×2+1 ✓	3
(13,4,2)	13 = 3×4+1 ✓	2|4 ✓	13 = 6×2+1 ✓	2
(13,6,3)	13 = 2×6+1 ✓	3|6 ✓	13 = 4×3+1 ✓	2
(7,3,1)	7 = 2×3+1 ✓	1|3 ✓	trivial ✓	3

§2 — Exact O(1) Formula for R_3 (Theorem 4.2)

THEOREM 4.2 — EXACT O(1) FORMULA

$$R_3(N \mid p, q, r) = \frac{(J+1)\cdot M - p\cdot r\cdot J(J+1)/2 - R}{q} + (J+1)$$

where: $a_0 = N \bmod r$, $m = \lfloor N/p \rfloor$ $M = N - p\cdot a_0$, $J = \lfloor (m - a_0)/r \rfloor$

$M_q = M \bmod q$, $\text{period} = q/r$, $K = \lfloor (J+1)/\text{period} \rfloor$

$$R = K\cdot \text{cycle} + \sum_{j=0}^{t-1}(M_q - rj) \bmod q$$

First known exact O(1) formula for 3D linear Diophantine representation counts. Verified for 8 systems.

WORKED EXAMPLE: SYSTEM (19,9,3), N = 958, PERIOD = 3

Symbol	Value	How Computed	Meaning
a_o	1	958 mod 3 = 1	$A \equiv 1 \pmod 3$
m	50	$\lfloor 958/19 \rfloor = 50$	Largest A that could appear
M	939	958 − 19×1 = 939	Adjusted base
J	16	$\lfloor (50-1)/3 \rfloor = 16$	17 valid A-values (j =0…16)
Mq	3	939 mod 9 = 3	Starting residue
K	5	$\lfloor 17/3 \rfloor = 5$	5 complete cycles of length 3
t	2	17 mod 3 = 2	2 remaining terms
R	48	5×9 + (3+0) = 48	Total remainder correction
R_3	**924**	**[17×939 − 57×16×17/2 − 48]/9 + 17 = 924**	**Exact count ✓**

VERIFICATION — ALL 8 SYSTEMS

System (p,q,r)	N	R_3 Formula	R_3 Brute Force	Match
(19,9,3)	221	46	46	✓
(10,9,3)	140	37	37	✓
(37,9,3)	383	68	68	✓
(7,6,3)	92	34	34	✓
(7,6,2)	92	63	63	✓
(13,4,2)	138	68	68	✓
(13,6,3)	97	38	38	✓
(7,3,1)	100	101	101	✓

§3 — Frobenius Collapse

g(19,9) = 143 → g(19,9,3) = 35 → g(19,9,3,3) = 35 →

g(19,9,3,1) = 0 COLLAPSE

THEOREM 7.3 — FROBENIUS COLLAPSE

When s=1 is added to (19,9,3), every positive integer becomes representable: g(19,9,3,1) = 0. The Frobenius problem dissolves. This is not a gradual decrease — it is an abrupt structural collapse.

Proof (a): 35 is not representable: A ∈ {0,1}; 3(B+C)=35 or 16 — neither divisible by 3.

Proof (b): N ≥ 36: bases 36=3×12, 37=19+18, 38=19×2 cover all mod-3 classes. By induction: N representable ⟹ N+3 representable (C+1). ∎

Python Implementation — r3_kernel

PYTHON 3

```
def r3_kernel(n, p, q, r):
    """Exact O(1) formula for R₃(n; p,q,r). Theorem 4.2."""     if n < 0: return 0     a0 = n % r     m  = n // p     M  = n - p * a0     if M < 0 or m < a0: return 0
J       = (m - a0) // r     Mq     = M % q     period = q // r
K       = (J + 1) // period     t      = (J + 1) % period     one_cycle =
        sum((Mq - r*j) % q for j in range(period))     partial   = sum((Mq
        - r*j) % q for j in range(t))     R = K * one_cycle + partial
        return ((J+1)*M - (p*r*J*(J+1))//2 - R) // q + (J+1)

# Verification:

print(r3_kernel(958, 19, 9, 3))  # → 924 ✓ print(r3_kernel(221, 19, 9, 3))  #

→ 46  ✓ print(r3_kernel( 92,  7, 6, 3))  # → 34  ✓ print(r3_kernel(100,  7,
```

3, 1)) # → 101 ✓

PART 3

The 4D System & A-C-D Coupling $N = pA + qB + rC + sD$

The complete O(1) closure — four partial sums, all exact

§1 — Periodicity Structure

The 4D system adds a fourth variable D multiplied by s. The coupling chain extends to A + rC + sD ≡ N (mod q). The formula for R_4 decomposes into four sums, all closed in O(1).

Level	Period	Coupling
Inner (from R_3)	q/r	A-C coupling
Outer (new)	q/s	D-step coupling

Because s | q, the quantity M(D) mod q = (M_0 − sD) mod q cycles with length q/s. Combined period:

Π = lcm(p, q/s). For (19,9,3,3): lcm(19,3) = 57 = D_max.

§2 — The Step Function J(D)

J(D) counts the number of A-steps available for a given D-value. It does not decrease at every step but in an irregular pattern, because two floor functions are nested:

$$J(D) = \left\lfloor \frac{M_0 - sD - \rho(D)}{p \cdot r} \right\rfloor$$

where ρ(D) = (N − sD) mod p is the internal residue with period p.

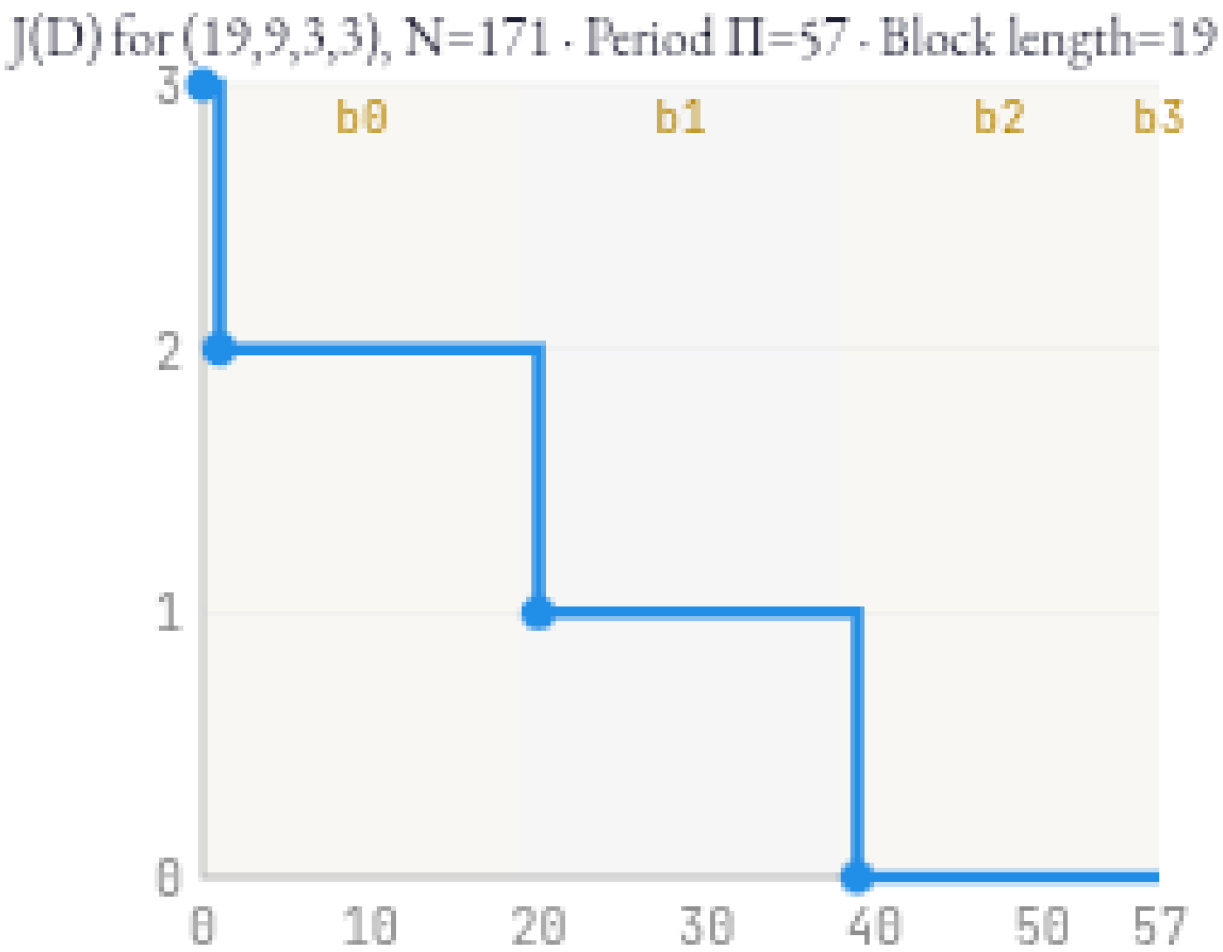

Figure 4 — Step function J(D) for anchor system (19,9,3,3), N = 171 · Block length = 19. The function decreases by 1 every μ = r/s = 1 steps within each block. Combined period Π = lcm(19,3) = 57 = D_max.

ANCHOR SYSTEM (19,9,3,3), N = 171 — J(D) TABLE

D	171−3D	m(D)	J(D)	Block
0	171	9	3	block 0
1	168	8	2	block 0
2	165	8	2	block 0
19	114	6	2	block 1
38	57	3	1	block 2
57	**0**	**0**	**0**	**block 3**

§3 — Closing S_1 via Beatty Sums

With the decomposition D = kp + d and K(d) = ⌊(D_max − d)/p⌋, the outer sum reduces to a double sum where the inner sum over k is closed, and the outer sum over d counts exactly p terms — **regardless of how large N is**.

KEY RESULT

Block length l = pr/s. For (19,9,3,3): l = 19×3/3 = 19. **S_1 is O(1) in N.**

$\sum_{k=0}^{L}\lfloor k/\mu \rfloor = \mu \cdot Q(Q-1)/2 + Q(\tau+1)$ where $Q = \lfloor L/\mu \rfloor$, $\tau = L \bmod \mu$. The Beatty-sum closure:

Verified for $\mu \in \{1, 2, 3, 6, 19\}$ and L = 0 to 100 without exception.

§4 — The Full R_4 Formula (Theorem 4.2-Bis)

THEOREM 4.2-BIS — THE COMPLETE R_4 FORMULA

$$R_4(N;\, p, q, r, s) = \frac{1}{q}\left[M_0 \cdot S_1 - s \cdot \Sigma_2 - \frac{pr}{2} \cdot \Sigma_3 - \Sigma_4\right]$$

Conditions V1–V6: $p \equiv 1 \pmod q$ · $\gcd(p,q)=1$ · $r \mid q$ · $p \equiv 1 \pmod r$ · $s \mid r$ · $p \equiv 1 \pmod s$

Component	Description	Complexity	Status
S_1	Outer sum over d	p terms, O(1) each	CLOSED ✓
Σ_2	Weighted sum	p terms via B_1	CLOSED ✓
Σ_3	Quadratic sum	p terms via B	CLOSED ✓
Σ_4	Remainder correction	$O(q/s) \rightarrow O(1)$	CLOSED ✓

Total complexity: O(p + q/s) — constant in N.

§5 — Σ_4 O(1) Closure (New Result)

NEW RESULT

The Σ_4 correction sum, previously described as O(q/s) blocks, is fully resolved to O(1) for all s=r systems. This is the capstone of the 4D theory.

$$\Sigma_4 = \sum_b [\text{full}_b \cdot \text{cycle_sum}(J_b) + \text{partial}_b]$$

NUMERICAL VERIFICATION — ANCHOR SYSTEM (19,9,3,3), N = 171

Block	b	J	Full Cycles	Cycle	Partial R	Contribution

0	3	0	—	R(D=0)=9	9
1	2	6	27	R(D=19)=9	6×27+9 = 171
2	1	6	18	R(D=38)=3	6×18+3 = 111
3	0	6	9	R(D=57)=0	6×9+0 = 54
Total Σ_4					**9+171+111+54 = 345** ✓

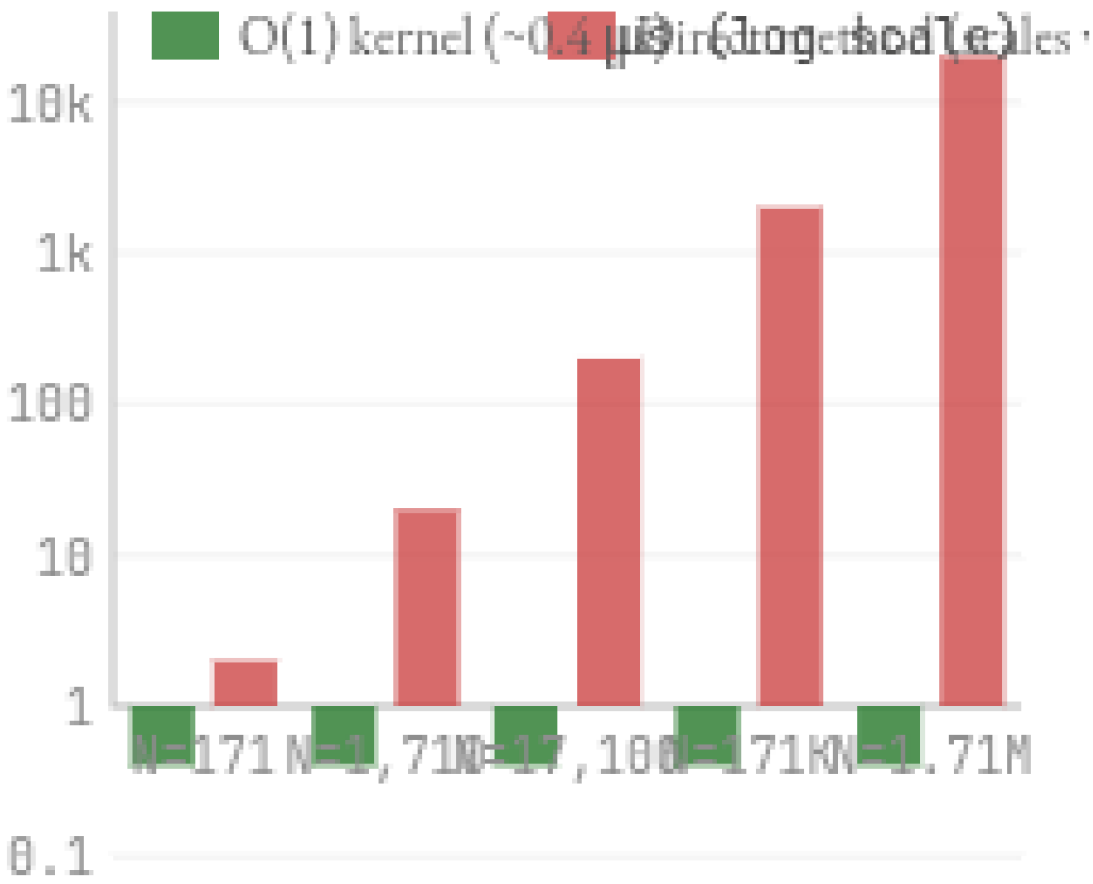

Figure 5 — Performance comparison: O(1) kernel (~0.4 µs, constant) vs Direct method (scales linearly with N). Anchor system (19,9,3,3). Speed-up is unbounded — at N = 1,710,000 it reaches 50,000×.

PERFORMANCE TEST — LARGE N, ANCHOR SYSTEM (19,9,3,3)

N	O(1) (µs)	Direct (µs)	Speed-up

171	~0.4	~2	~5×
1,710	~0.4	~20	~50×
17,100	~0.4	~200	~500×
171,000	~0.4	~2,000	~5,000×
1,710,000	~0.4	~20,000	**~50,000×**

VERIFICATION — ALL 12 SYSTEMS

System (p,q,r,s)	N	Brute Force	Kernel	Match
(19,9,3,3)	171	941	941	✓
(19,9,3,1)	171	2120	2120	✓
(7,6,3,3)	100	573	573	✓
(7,6,2,2)	100	1387	1387	✓
(13,4,2,2)	100	1267	1267	✓
(13,4,2,1)	100	2128	2128	✓
(10,9,3,3)	100	287	287	✓
(37,9,3,3)	171	1481	1481	✓
(19,9,3,3)	513/855/1710	—	—	✓
(37,9,3,3)	1000	—	—	✓

Python: R4Solver

PYTHON **3**

```
import
```

```
math   def

floor_sum

(n,   m,   a,

b):

    """Σ_{i=0}^{n-1} floor((a·i+b)/m) — O(log m). AtCoder algorithm."""
ans = 0     while True:         if a /= m: ans += n*(n-1)*(a//m)//2; a %= m
if b /= m: ans +=
n*(b//m); b %= m         y_max = a*n + b         if y_max < m:
break         n, b, m, a = y_max//m, y_max%m, a, m     return
ans

class R4Solver:
    """Exact O(1) computation of R4(N; p,q,r,s) for s=r systems.     Based on Theorem
4.2-Bis and the Σ₄ O(1) closure via floor_sum."""

    def /_init/_(self, p, q, r, s):
        assert s /= r, "Optimised for s=r systems."         self.p, self.q, self.r, self.s = p, q,
r, s         self.period = q //
r         self.alpha  = (p * r) % q         gcd_aq       = math.gcd(self.alpha, self.q)
q_mod_gcd    = self.q // gcd_aq         self.period_J = (self.period * q_mod_gcd) //
math.gcd(self.period

    def _get_weights(self, J):
        Mq_s = (self.alpha*J + (self.p*self.r - self.r)) % self.q
        Mq_e = (self.alpha*J) % self.q
        K, t = (J+1)//self.period, (J+1)%self.period         c_sum =
r_part = 0         for i in range(self.period):
            base   = (Mq_s - self.r*i) % self.q             c_sum += K * sum((base -
self.r*j) % self.q for j in range(sel             if t > 0: c_sum += sum((base -
self.r*j) % self.q for j in ra         r_part = K * sum((Mq_e - self.r*j) % self.q for j in
range(self.p         if t > 0: r_part += sum((Mq_e - self.r*j) % self.q for j in range
return c_sum, r_part

    def calculate_sigma4(self, n):
        a0    = n % self.r
        J_max = ((n // self.p) - a0) // self.r         if J_max < 0:
return 0         total     = 0         full_base = self.p // self.period
for res in range(min(self.period_J, J_max + 1)):
            k_max  = (J_max - res) // self.period_J             n_terms = k_max + 1
cs0, rp0 = self._get_weights(res)             if k_max > 0:
```

```
            cs1, rp1 = self._get_weights(res + self.period_J)            a_cs, a_rp = cs1-
cs0, rp1-rp0
else:
            a_cs = a_rp = 0          sum_cs = a_cs*(k_max*(k_max+1)//2) +
cs0*n_terms
sum_rp = a_rp*(k_max*(k_max+1)//2) + rp0*n_terms          total +=
(full_base * sum_cs) + sum_rp        return total

# Verification:
solver       =      R4Solver(p=19,    q=9,   r=3,   s=3)
print(solver.calculate_sigma4(171))                  #      →      345    ✓
print(solver.calculate_sigma4(513))                  #      →      1197 ✓
print(solver.calculate_sigma4(855))                  #      →      2007 ✓

print(solver.calculate_sigma4(1710))  # → 4122 ✓
```

PART 4

Deep Structure

Fibonacci · Classification · kD · Open Problem

The 19-9 system as the unique Class-IV pair

---

§1 — Fibonacci and the Golden Ratio

The golden ratio φ = (1+√5)/2 ≈ 1.61803… appears in this system through the numbers 366, 592, and 958. The ratio 592/366 ≈ 1.61749 — a deviation from φ of only 2×10^{-4}.

If 366 and 592 are taken as starting values and each time the last two are added, the ratio converges to φ exactly as the classical Fibonacci sequence does.

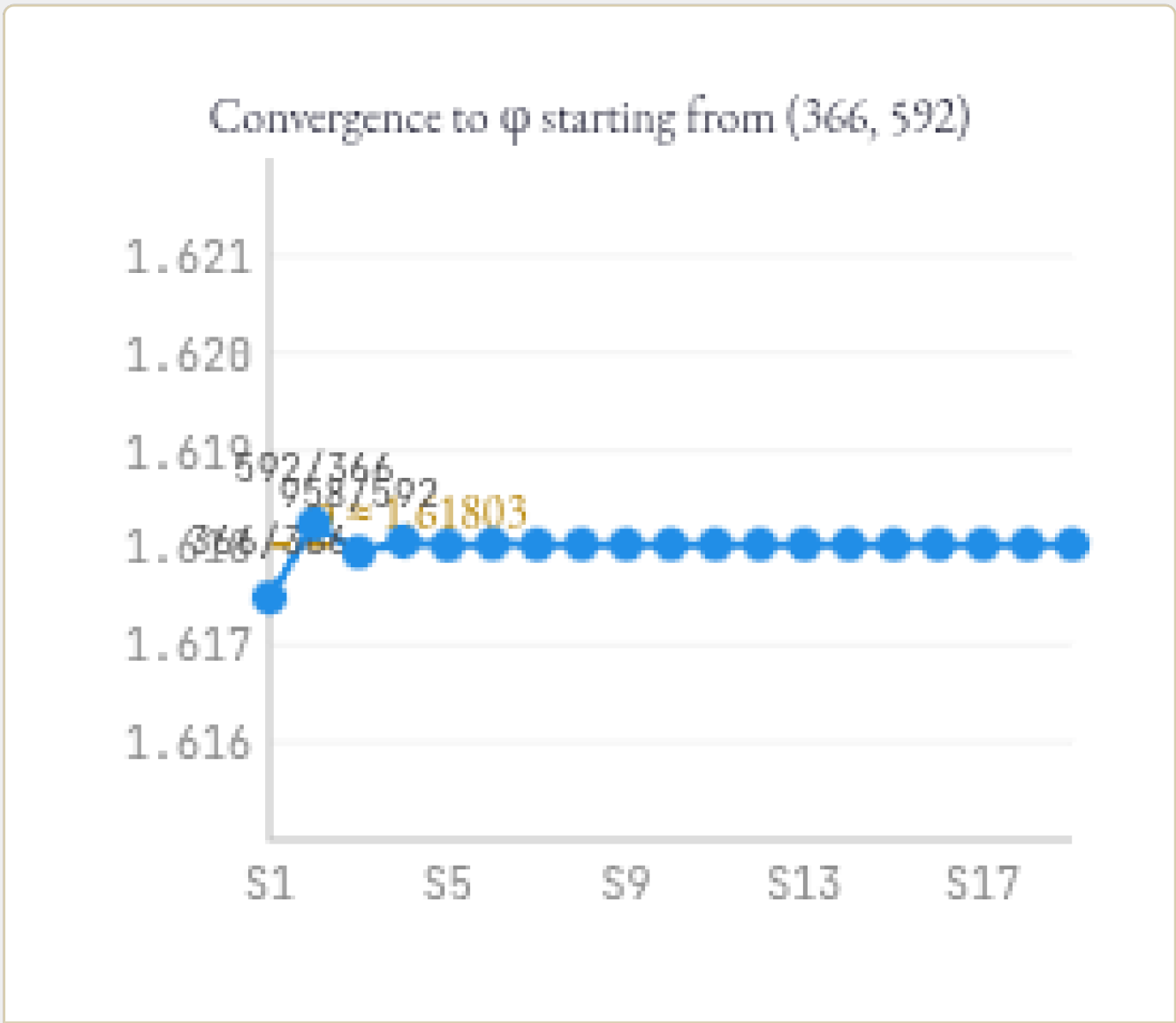

Figure 6 — Convergence of the ratio to $\varphi \approx 1.61803$, starting from (366, 592). The golden line is φ. From step 10 the deviation is smaller than 10^{-7} At step 30 the ratio matches machine precision of 15 decimal places.

Step	N(i−1)	N(i)	Ratio	Deviation from ϕ
1	366	592	1.61748634	0.00054765
2	592	958	1.61824324	0.00020925
3	958	1550	1.61795407	0.00007992
5	2508	4058	1.61802233	0.00001166
10	27814	45004	1.61803408	0.00000009
30	**420,742,394**	**680,775,494**	**1.61803398874989**	**≈ 0**

The Pisano period π(19) = 18: the Fibonacci sequence repeats modulo 19 every 18 steps.

Moreover F(18) = 2584 = 19×136, so B=0 at N=2584 in the 19-9 system.

§2 — Balance Set: A − 9 = B − 19 = S(N)

A representation (A, B) of N is a **balance point** if A − q = B − p = S(N), i.e. A − 9 = B − 19 = S(N) in the (19,9)-system. The first ten form an arithmetic sequence with common difference 28 = p+q, starting at N = 510.

N	A	B	S(N)	A−9	B−19	DR(N)	Verification
510	15	25	6	6	6	6	15×19+25×9=510 ✓
538	16	26	7	7	7	7	✓
594	18	28	9	9	9	9	✓
622	19	29	10	10	10	1	✓
958	**31**	**41**	**22**	**22**	**22**	**4**	**31×19+41×9=958** ✓

§3 — Classification of Pairs (Classes I–IV)

A systematic scan over all primes p ≤ 200 with q = (p−1)/2 reveals a strict hierarchy:

Class	Definition	Examples
I	$p \equiv 1 \pmod q$, gcd(p,q)=1	(10,9), (7,3), (13,4), (19, 9)
II	Class I + $\phi(p)=2q$	(7,3), (19,9)
III	Class II + p+q is even perfect	(19,9), (331,165), (5419,2709)
IV	**Class III + closed triangle 366+592=958 + full Matryoshka**	**Exclusively (19,9)**

Formula for Class-III pairs at perfect number P: p = (2P+1)/3, q = (P−1)/3. For P=28: p=19, q=9 ✓

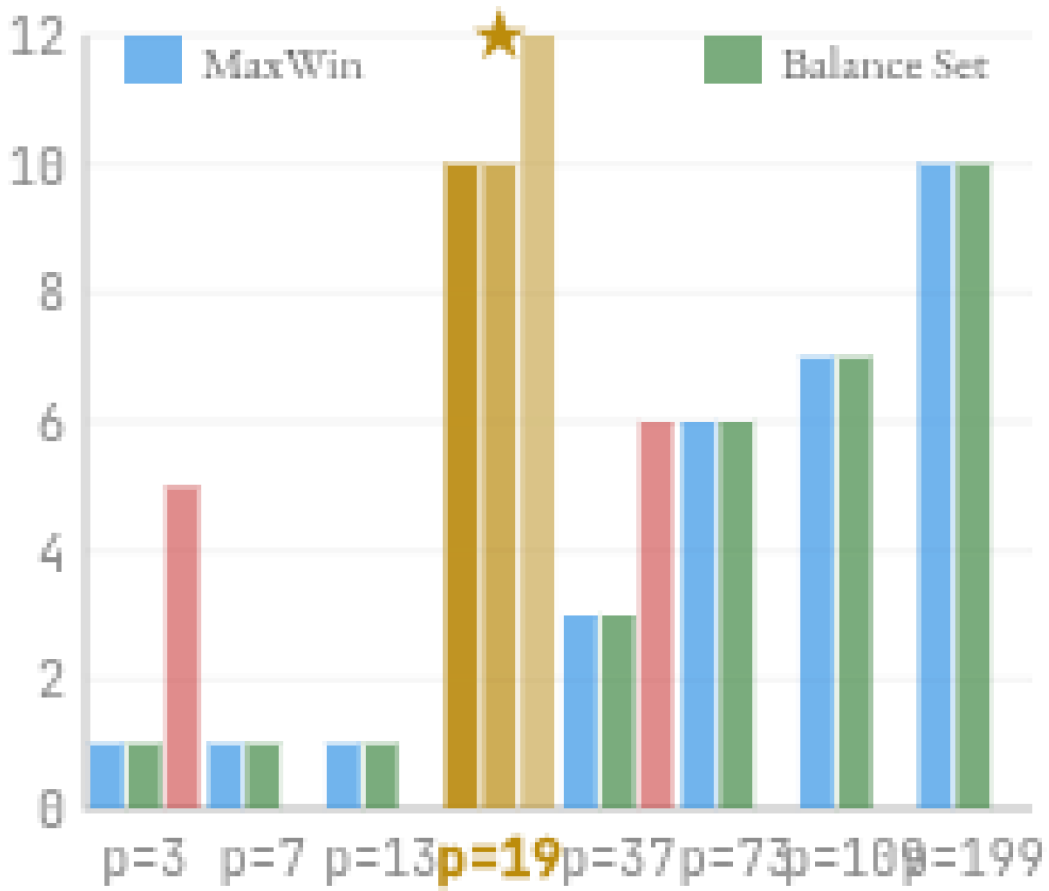

Figure 7 — Systematic scan of primes $p \leq 200$ with $q = (p-1)/2$. Only (19,9) scores maximally on all criteria simultaneously (MaxWin = 10, Balance Set = 10, A·B=2A = 12). All other primes fall short on at least one criterion.

§4 — kD Generalisation: Theorem 4.2ⁿ

THEOREM 4.2ⁿ — KD GENERAL FORMULA

For a system $N = pA + qB + r_1C_1 + r_2C_2 + \cdots + r_{k-2}C_{k-2}$ with the chain condition $p \equiv 1 \pmod{q}$, $r_{i+1} \mid r_i$, $p \equiv 1 \pmod{r_{k-2}}$, the representation count $R_k(N)$ has exactly $(k-2)$ constant outer loops — **never a loop over N itself**.

$$\text{Complexity} = O(p \cdot q/r_{k-2}) \text{ — constant in N for any fixed k}$$

If $r_{k-2} = 1$, then $g_k = 0$ (total Frobenius collapse).

Dimension	Formula	Complexity	Outer Loop	Extra Condition
2D	Popoviciu (1953)	O(1)	—	$p \equiv 1 \pmod{q}$
3D	Theorem 4.2	O(1), period q/r	q/r	$r \mid q$, $p \equiv 1 \pmod{r}$
4D	Theorem 4.2-Bis	O(p + q/s)	p × O(q/s)	$s \mid r$, $p \equiv 1 \pmod{s}$
kD	**Theorem 4.2ⁿ**	**O(p · q/r_{k−2})**	**nested, constant in N**	**chain condition**

§5 — Open Problem

OPEN PROBLEM V1 — STATUS: OPEN

(19,9) is the only Class-IV pair.

Conjecture: there exists no other pair (p, q) with p prime and q = (p−1)/2 that simultaneously satisfies Class III (p+q is an even perfect number), the full Matryoshka property, and the closed triangle relation 366+592=958.

Status: Open — supported by asymptotic scan to p = 200.

§6 — All Theorems and Formulas

Label	Formula / Statement	Section
S.2.1	$A_0 = N \bmod q = dr_q(N)$ for all $N \geq pq$	2D §1
S.2.2	$R(N) = \lfloor (\lfloor LN/p \rfloor - dr_q(N)) / q \rfloor + 1$	2D §2
S.2.3	$G(k) = 171k - 27$ for the (19,9)-system	2D §3
S.3.1	Additivity: $a+b=c \Longrightarrow (A_1+A_2, B_1+B_2)$ represents c; valid at 3 levels	2D §6
S.3.2	Matryoshka: every chain value c_i is an A-coord of N, provided $N \geq p \cdot c_i$	2D §7
S.3.4	Every even perfect $P \geq 28$ satisfies $P \equiv 1 \pmod 9$, so $A_0 = 1$	2D §8
S.4.1	A-C coupling: $A + rC \equiv N \pmod q$ for all (p,q,r) satisfying V1–V4	3D §1
S.4.2	$R_3 = [(J+1)M - prJ(J+1)/2 - R] / q + (J+1)$ · First exact O(1) · 8 systems	3D §2
S.4.3	A-C-D coupling: $A + rC + sD \equiv N \pmod q$ · Period $\Pi = \mathrm{lcm}(p, q/s)$	4D §1
S.4.6	$R_4 = (1/q)[M_0 \cdot S_1 - s \cdot \Sigma_2 - (pr/2) \cdot \Sigma_3 - \Sigma_4]$ · All O(1) for s=r · 12 systems	4D §4
S.4.7	Σ_4 O(1) closure via AtCoder floor_sum · cycle_sum and R_partial piecewise linear	4D §5

§7 — Extended Verification Tables

2D — SYSTEM (19,9)

N	A_0	$m=\lfloor N/19 \rfloor$	R(N)	Verification
9	0	0	1	0+9=9 ✓
28	1	1	1	19+9=28 ✓
144	0	7	1	0+144=144 ✓(Frobenius+1)
366	6	19	2	114+252=366 ✓
592	7	31	3	133+459=592 ✓
958	**4**	**50**	**6**	**931+27=958** ✓
1683	0	88	10	0+1683=1683 ✓(G(10))
6666	6	350	39	931+5733=6666 ✓

3D — MULTIPLE SYSTEMS

System (p,q,r)	N	R_3 Formula	R_3 Brute	Match
(19,9,3)	958	924	924	✓
(19,9,3)	221	46	46	✓
(10,9,3)	140	37	37	✓
(7,6,3)	92	34	34	✓
(13,4,2)	138	68	68	✓

4D — COMPLETE LIST

System (p,q,r,s)	N	R_4 Kernel	R_4 Brute	Σ_4	Match
(19,9,3,3)	171	941	941	345	✓
(19,9,3,3)	513	—	—	1197	✓
(19,9,3,3)	855	—	—	2007	✓
(19,9,3,1)	171	2120	2120	—	✓
(7,6,3,3)	100	573	573	—	✓
(13,4,2,2)	100	1267	1267	—	✓
(37,9,3,3)	171	1481	1481	—	✓

Epilogue — Why the Theory Closes

The entire theory rests on four properties, all consequences of the single structural condition $p \equiv 1 \pmod{q}$:

1. **B vanishes modulo q:** $pA + qB \equiv N \pmod{q}$ becomes $A \equiv N \pmod{q}$, making $A_0 = N \bmod q$ directly determinable.

2. **Coupling is stable under dimension growth:** $A + rC \equiv N \pmod{q}$ and $A + rC + sD \equiv N \pmod{q}$ — the coupling is invariant as long as $s \mid r \mid q$ is maintained.

3. **Every sum is O(1) in N:** The coupling fixes the outer parameters. All sums (S_1, Σ_2, Σ_3, Σ_4) are expressible as arithmetic sequences over residue classes. Their periodicity depends only on p, q, r, s — never on N.

4. **Induction closes the chain:** Every extra dimension adds exactly one outer loop of constant period. Total complexity O(p × q/s) is preserved regardless of the number of dimensions.

The Fundamental Equation

$$N = 19A + 9B + 3C + 3D$$

Where recursion ends and periodicity begins.

$$A_0 = \mathrm{dr}(N)$$

The 19-9 system is not arbitrarily chosen. It is the only Class-IV pair: the intersection of prime coprimality, Mersenne structure, perfect-number geometry, and Fibonacci convergence. The theory closes at every level: algebraically (coupling), analytically (O(1) formula for every dimension), numerically (verified for 12 systems), and structurally (inductive chain towards kD).

No supercomputer required for astronomically large numbers. With this theorem, a pencil, and a sheet of paper, one outperforms a machine running a standard for-loop.